

Kummer function and High energy String Scatterings

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Abstract

Based on a summation algorithm for Stirling number identity developed recently, we discover that the ratios calculated previously among high energy string scattering amplitudes in the Gross regime (GR) can be extracted from the Kummer function of the second kind. This function naturally shows up in the leading order of high energy string scattering amplitudes in the Regge regime (RR). As a result, the identity suggested by string theory calculation can be rigorously proved by a totally different but sophisticated mathematical method. We conjecture and give evidences that the existence of these ratios in the RR persists to all orders in the Regge expansion of high energy string scattering amplitudes.

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High energy limits of scattering amplitudes are of fundamental importance in quantum mechanics, quantum field theory and string theory. Not only can they be used to greatly simplify a lot of mathematical calculations of the amplitudes but also that one can use the high energy amplitudes to extract many fundamental characteristics of the physical theory. There are two fundamental regimes of high energy scattering amplitudes, namely, the fixed angle regime and the fixed momentum transfer regime. These two regimes represent two different high energy perturbation expansions of the scattering amplitudes, and contain complementary information of the underlying theory. In QCD, for example, the probe of high energy, fixed angle regime reveals the partonic structures of hadrons, quarks and gluons. On the other hand, the Regge behavior of high energy hadronic scattering amplitudes suggested a string model of hadrons with a linear relation between hadron spins and their mass squared. In string theory, the scattering amplitudes in the high energy, fixed angle regime [1, 2, 3], the Gross regime (GR), were recently intensively reinvestigated for massive string states at arbitrary mass levels [4, 5, 6, 7, 8, 9, 10]. See also [11, 12, 13]. An infinite number of linear relations, or stringy symmetries, among string scattering amplitudes of different string states were obtained. Moreover, these linear relations can be solved for each fixed mass level $M^2 = 2(n-1)$, and ratios $T^{(n,2m,q)}/T^{(n,0,0)}$, $n \geq 2m+2q$; $m, q \geq 0$ among the amplitudes can be obtained. An important new ingredient of these calculations is the decoupling of zero-norm states (ZNS) [14, 15, 16] in the old covariant first quantized (OCFQ) string spectrum. Since so far there does not exist any algebraic structure (or group structure) of this high energy 26D spacetime symmetry (except ω_∞ for the case of toy 2D string theory [15]), mathematically the meaning of these infinite number of ratios remains mysterious.

In this letter, we calculate high energy massive string scattering amplitudes in the fixed momentum transfer regime, the Regge regime (RR). There have been some previous studies [17, 18, 19, 20, 21, 22] of high energy string scatterings in this regime in the literature. Our motivation here is to calculate massive string scatterings and try to extract possible patterns of the scattering amplitudes in the RR which may mimic the patterns (e.g. the ratios mentioned above) in the GR. Since the decoupling of ZNS applies to all kinematic regime, it is reasonable to expect some implication of this decoupling in the RR. We found that [23] the number of high energy scattering amplitudes for each fixed mass level in the RR is much more numerous than that of GR calculated previously. On the other hand, it

seems that both the saddle-point method and the method of decoupling of high energy ZNS adopted in the calculation of GR do not apply to the case of RR. However the calculation is still manageable, and the general formula for the high energy scattering amplitudes for each fixed mass level in the RR can be written down explicitly [23]. As expected, there is no linear relation anymore as in the case of scatterings in the GR. Moreover, we discover that the leading order amplitudes at each fixed mass level in the RR can be expressed in terms of the Kummer function of the second kind. More surprisingly, for those leading order high energy amplitudes $A^{(n,2m,q)}$ in the RR with the same type of $(n, 2m, q)$ as those of GR, we can extract from them the above mentioned ratios $T^{(n,2m,q)}/T^{(n,0,0)}$ in the GR by using Kummer function of the second kind, which naturally shows up in the leading order of high energy string scattering amplitudes in the RR. The calculation brings a link between high energy string scattering amplitudes in the GR and the RR. In addition to the decoupling of ZNS calculated previously [4, 5, 6, 7, 8, 9, 10] (or the unitarity of quantum string theory), the calculation from Kummer function in this letter seems to give another interpretation of the existence of these ratios. Finally, we calculate some Regge string scattering amplitudes to subleading orders and conjecture that these ratios persist to all orders in the Regge expansion of high energy string scattering amplitudes.

We stress that, mathematically, the proof of the identification of the ratios in the GR from the Kummer function calculated in the RR turns out to be highly nontrivial. This is based on a summation algorithm for Stirling number identity derived by Mkauers in 2007 [24]. It is very interesting to see that the identity in Eq.(17) suggested by string theory calculation can be rigorously proved by a totally different but sophisticated mathematical method. Although this kind of coincidence is not unusual in the development of string theory, our results bring an interesting connection between string theory and combinatoric number theory. Moreover, the connection between Kummer function and high energy string scatterings may shed light on a deeper understanding of stringy symmetries.

We begin with a brief review of high energy string scatterings in the GR. That is in the kinematic regime $s, -t \rightarrow \infty$, $t/s \approx -\sin^2 \frac{\theta}{2}$ fixed (but $\theta \neq 0$) where s, t and u are the Mandelstam variables and θ is the CM scattering angle. It was shown [7, 8] that for the 26D open bosonic string the only states that will survive the high-energy limit at mass level

$M_2^2 = 2(n-1)$ are of the form

$$|n, 2m, q\rangle \equiv (\alpha_{-1}^T)^{n-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q |0, k_2\rangle \quad (1)$$

where the polarizations of the 2nd particle with momentum k_2 on the scattering plane were defined to be $e^P = \frac{1}{M_2}(E_2, k_2, 0) = \frac{k_2}{M_2}$ as the momentum polarization, $e^L = \frac{1}{M_2}(k_2, E_2, 0)$ the longitudinal polarization and $e^T = (0, 0, 1)$ the transverse polarization. Note that e^P approaches to e^L in the GR, and the scattering plane is defined by the spatial components of e^L and e^T . Polarizations perpendicular to the scattering plane are ignored because they are kinematically suppressed for four point scatterings in the high-energy limit. One can then use the saddle-point method to calculate the high energy scattering amplitudes. For simplicity, we choose k_1 , k_3 and k_4 to be tachyons and the final result of the ratios of high energy, fixed angle string scattering amplitude are [7, 8]

$$\frac{T^{(n, 2m, q)}}{T^{(n, 0, 0)}} = \left(-\frac{1}{M_2}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!! \quad (2)$$

The ratios in Eq.(2) can also be obtained by using the decoupling of two types of ZNS in the spectrum. As an example, for $M_2^2 = 4$ we get [4, 5]

$$T_{TTT} : T_{LLT} : T_{(LT)} : T_{[LT]} = 8 : 1 : -1 : -1. \quad (3)$$

To convince the readers that the infinite ratios in Eq.(2) are the symmetries or, at least, remnant of full spacetime symmetries of 26D string theory, it was shown that a set of 2D discrete ZNS Ω_{J_1, M_1}^+ carry ω_∞ symmetry charges [15]

$$\int \frac{dz}{2\pi i} \Omega_{J_1, M_1}^+(z) \Omega_{J_2, M_2}^+(0) = (J_2 M_1 - J_1 M_2) \Omega_{(J_1+J_2-1), (M_1+M_2)}^+(0). \quad (4)$$

A natural question arises. Is there any mathematical structure (e.g. group structure) of these infinite number of ratios? Let's consider a simple analogy from particle physics. The ratios of the nucleon-nucleon scattering processes

$$\begin{aligned} (a) \quad & p + p \rightarrow d + \pi^+, \\ (b) \quad & p + n \rightarrow d + \pi^0, \\ (c) \quad & n + n \rightarrow d + \pi^- \end{aligned} \quad (5)$$

can be calculated to be

$$T_a : T_b : T_c = 1 : \frac{1}{\sqrt{2}} : 1 \quad (6)$$

from $SU(2)$ isospin symmetry. Similarly, as we will see in the rest of the paper, the ratios in Eq.(2) can be extracted from Kummer function. The key is to study high energy string scatterings in the RR.

We now turn to the discussion on high energy string scatterings in the RR. That is in the kinematic regime $s \rightarrow \infty$, $\sqrt{-t}$ fixed (but $t \neq -\infty$). It can be shown [23] that the number of high energy scattering amplitudes for each fixed mass level in the RR is much more numerous than those calculated from Eq.(1) in the GR. For our purpose here, however, we will only calculate scattering amplitudes corresponding to the vertex in Eq.(1). The relevant kinematics are

$$e^P \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^P \cdot k_3 \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t - M_2^2 - M_3^2}{2M_2}; \quad (7)$$

$$e^L \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^L \cdot k_3 \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t + M_2^2 - M_3^2}{2M_2}; \quad (8)$$

and

$$e^T \cdot k_1 = 0, \quad e^T \cdot k_3 \simeq -\sqrt{-t}. \quad (9)$$

Note that e^P *does not* approach to e^L in the RR. The Regge scattering amplitude for the $s - t$ channel can be calculated to be

$$\begin{aligned} A^{(n,2m,q)} &= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{e^T \cdot k_3}{1-x} \right]^{n-2m-2q} \\ &\quad \left[\frac{e^L \cdot k_1}{-x} + \frac{e^L \cdot k_3}{1-x} \right]^{2m} \left[\frac{e^L \cdot k_1}{x^2} + \frac{e^L \cdot k_3}{(1-x)^2} \right]^q \\ &\simeq (\sqrt{-t})^{n-2m-2q} \left(\frac{\tilde{t}'}{2M_2} \right)^q \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3 - n + 2m} \\ &\quad \sum_{j=0}^{2m} \binom{2m}{j} \left(\frac{s}{2M_2 x} \right)^j \left(\frac{-\tilde{t}'}{2M_2(1-x)} \right)^{2m-j} \\ &= (\sqrt{-t})^{n-2m-2q} \left(\frac{\tilde{t}'}{2M_2} \right)^q \left(\frac{\tilde{t}'}{2M_2} \right)^{2m} \\ &\quad \sum_{j=0}^{2m} \binom{2m}{j} (-1)^j \left(\frac{s}{\tilde{t}'} \right)^j B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 - n + j + 1). \end{aligned} \quad (10)$$

Note that the term $\frac{e^L \cdot k_1}{x^2}$ in the bracket is subleading in energy and can be neglected. In the high energy limit, the beta function in Eq.(10) can be approximated by

$$B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 - n + j + 1) \simeq B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{-j} \left(-1 - \frac{t}{2}\right)_j \quad (11)$$

where $(a)_j = a(a+1)(a+2)\dots(a+j-1)$ is the Pochhammer symbol. Finally, the leading order amplitude in the RR can be written as

$$A^{(n,2m,q)} = B \left(-1 - \frac{s}{2}, -1 - \frac{t}{2} \right) \sqrt{-t}^{n-2m-2q} \left(\frac{1}{2M_2} \right)^{2m+q} 2^{2m} (\tilde{t}')^q U \left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}'}{2} \right), \quad (12)$$

which is UV power-law behaved as expected. U in Eq.(12) is the Kummer function of the second kind and is defined to be

$$U(a, c, x) = \frac{\pi}{\sin \pi c} \left[\frac{M(a, c, x)}{(a-c)!(c-1)!} - \frac{x^{1-c} M(a+1-c, 2-c, x)}{(a-1)!(1-c)!} \right] \quad (c \neq 2, 3, 4, \dots) \quad (13)$$

where $M(a, c, x) = \sum_{j=0}^{\infty} \frac{(a)_j x^j}{(c)_j j!}$ is the Kummer function of the first kind. U and M are the two solutions of the Kummer equation

$$xy''(x) + (c-x)y'(x) - ay(x) = 0. \quad (14)$$

At this point, it is crucial to note that, in our case of Eq.(12), $c = \frac{t}{2} + 2 - 2m$ and is not a constant as in the usual definition, so U in Eq.(12) is *not* a solution of the Kummer equation. This will make our follow-up analysis, the proof of Eq.(17) discussed below, more complicated as we will see soon. On the contrary, since $a = -2m$ an integer, the Kummer function in Eq.(12) terminated to be a finite sum (see Eq.(15) below). This will simplify the manipulation of Kummer function used in this paper. We stress that all the calculations in this paper do not rely on Kummer equation Eq.(14). In fact, one can take Eq.(15) below as a formal definition of Kummer function used in this paper.

It is important to note that there is no linear relation among high energy string scattering amplitudes of different string states for each fixed mass level in the RR as can be seen from Eq.(12). This is very different from the result in the GR in Eq.(2). In other words, the ratios $A^{(n,2m,q)}/A^{(n,0,0)}$ are t -dependent functions. In particular, we can extract the coefficients of the highest power of t in $A^{(n,2m,q)}/A^{(n,0,0)}$. We can use the identity of the Kummer function

$$\begin{aligned} 2^{2m} (\tilde{t}')^{-2m} U \left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}'}{2} \right) &= {}_2F_0 \left(-2m, -1 - \frac{t}{2}, -\frac{2}{\tilde{t}'} \right) \\ &\equiv \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2} \right)_j \frac{\left(-\frac{2}{\tilde{t}'} \right)^j}{j!} = \sum_{j=0}^{2m} \binom{2m}{j} \left(-1 - \frac{t}{2} \right)_j \left(\frac{2}{\tilde{t}'} \right)^j \end{aligned} \quad (15)$$

to calculate

$$\frac{A^{(n,2m,q)}}{A^{(n,0,0)}} = (-1)^q \left(\frac{1}{2M_2}\right)^{2m+q} (-t)^m \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2}\right)_j \frac{(-2/t)^j}{j!} + O\left\{\left(\frac{1}{t}\right)^{m+1}\right\} \quad (16)$$

where we have replaced \tilde{t}' by t as t is large. *If the leading order coefficients in Eq.(16) extracted from the high energy string scattering amplitudes in the RR are to be identified with the ratios calculated previously among high energy string scattering amplitudes in the GR in Eq.(2), we need the following identity*

$$\begin{aligned} & \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2}\right)_j \frac{(-2/t)^j}{j!} \\ &= 0(-t)^0 + 0(-t)^{-1} + \dots + 0(-t)^{-m+1} + \frac{(2m)!}{m!} (-t)^{-m} + O\left\{\left(\frac{1}{t}\right)^{m+1}\right\}. \end{aligned} \quad (17)$$

The coefficient of the term $O\{(1/t)^{m+1}\}$ in Eq.(17) is irrelevant for our discussion. The proof of Eq.(17) turns out to be nontrivial. The standard approach by using integral representation of the Kummer function seems not applicable here. Presumably, the difficulty of the rigorous proof of Eq.(17) is associated with the unusual non-constant c in the argument of Kummer function in Eqs.(12) and (15) as mentioned above. It is a nontrivial task to do the proof compared to the usual cases where the argument c of the Kummer function is a constant. Here we will adopt another approach to prove Eq.(17). This approach strongly relies on the algorithm for Stirling number identity derived by Mkauers [24] in 2007, and is highly nontrivial either. The leading order identity of Eq.(17) can be written as

$$f(m) \equiv \sum_{j=0}^m (-1)^j \binom{2m}{j+m} [s(j+m-1, j-1) + s(j+m-1, j)] = (2m-1)!! \quad (18)$$

where the signed first Stirling number $s(n, k)$ is defined to be

$$(x)_n = \sum_{k=0}^n (-1)^{n-k} s(n, k) x^k. \quad (19)$$

The authors had verified the validity of Eq.(18) for $m = 1, 2, \dots, 2000$ before they carried out the exact proof to be discussed below. To prove Eq.(18) we define

$$f(u, m) \equiv \sum_{j=0}^{m+u} (-1)^j \binom{2m+u}{j+m} [s(j+m-1, j-1) + s(j+m-1, j)] \quad (20)$$

with $f(0, m) = f(m)$. By using the result of [24], one can prove that $f(u, m)$ satisfies the following recurrence relation

$$-(1 + 2m + u)f(u, m) + (2m + u)f(u + 1, m) + f(u, m + 1) = 0. \quad (21)$$

Eq.(21) is the most nontrivial step to prove Eq.(18). Finally by taking $u = 0$, it can be shown that the second term of Eq.(21) vanishes [23]. Eq.(18) is then proved by mathematical induction. The vanishing of the coefficients of $(-t)^0, (-t)^{-1}, \dots, (-t)^{-m+1}$ terms on the LHS of Eq.(17) means, for $1 \leq i \leq m$,

$$g(m, i) \equiv \sum_{j=0}^{m+i} (-1)^{j-i} \binom{2m}{j+m-i} [s(j+m-1-i, j) + s(j+m-1-i, j-1)] = 0. \quad (22)$$

To prove this identity, we need the recurrence relation [24]

$$\begin{aligned} & -2(1+m)^2(1+2m)g(m, i) + (2+7m+4m^2)g(m+1, i) \\ & -2m(1+m)(1+2m)g(m+1, i+1) - m \times g(m+2, i) = 0. \end{aligned} \quad (23)$$

Putting $i = 0, 1, 2, \dots$, and using the fact we have just proved, i.e. $g(m+1, 0) = (2m+1)g(m, 0)$, one can prove Eq.(22). Eq.(17) is thus finally proved. It is very interesting to see that the identity in Eq.(17) suggested by string scattering amplitude calculation can be rigorously proved by a totally different but sophisticated mathematical method. In conclusion, ratios in Eq.(2) can be extracted from Kummer function of the second kind

$$\frac{T(n, 2m, q)}{T(n, 0, 0)} = \left(-\frac{1}{2M}\right)^{2m+q} 2^{2m} \lim_{t \rightarrow \infty} (-t)^{-m} U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{t}{2}\right). \quad (24)$$

In view of Eq.(6), this result may help to uncover the fundamental symmetry of string theory. At last, we give an explicit calculation of the high energy string scattering amplitudes to subleading orders in the RR for $M_2^2 = 4$ [23]

$$A_{TTT} \sim \frac{1}{8} \sqrt{-t} t s^3 + \frac{3}{16} \sqrt{-t} t(t+6) s^2 + \frac{3t^3 + 84t^2 - 68t - 864}{64} \sqrt{-t} s + O(1), \quad (25)$$

$$\begin{aligned} A_{LLT} \sim & \frac{1}{64} \sqrt{-t} (t-6) s^3 + \frac{3}{128} \sqrt{-t} (t^2 - 20t - 12) s^2 \\ & + \frac{3t^3 - 342t^2 - 92t + 5016 + 1728(-t)^{-1/2}}{512} \sqrt{-t} s + O(1), \end{aligned} \quad (26)$$

$$\begin{aligned} A_{(LT)} \sim & -\frac{1}{64} \sqrt{-t} (t+10) s^3 - \frac{1}{128} \sqrt{-t} (3t^2 + 52t + 60) s^2 \\ & - \frac{3[t^3 + 30t^2 + 76t - 1080 - 960(-t)^{-1/2}]}{512} \sqrt{-t} s + O(1), \end{aligned} \quad (27)$$

$$A_{[LT]} \sim -\frac{1}{64}\sqrt{-t}(t+2)s^3 - \frac{3}{128}\sqrt{-t}(t+2)^2s^2 - \frac{(3t-8)(t+6)^2[1-2(-t)^{-1/2}]}{512}\sqrt{-t}s + O(1). \quad (28)$$

We have ignored an overall irrelevant factors in the above amplitudes. Note that the calculation of Eq.(27) and Eq.(28) involves amplitude of the state $(\alpha_{-2}^T)(\alpha_{-1}^L)|0, k_2\rangle$ which can be shown to be of leading order in the RR [23], but is of subleading order in the GR as it is not in the form of Eq.(1). However, the contribution of the amplitude calculated from this state will not affect the ratios $8 : 1 : -1 : -1$ in the RR [23]. One can now easily see that the ratios of the coefficients of the highest power of t in these leading order coefficient functions $\frac{1}{8} : \frac{1}{64} : -\frac{1}{64} : -\frac{1}{64}$ agree with the ratios in the GR calculated in Eq.(3) as expected. Moreover, one further observation is that these ratios remain the same for the coefficients of the highest power of t in the subleading orders $(s^2) \frac{3}{16} : \frac{3}{128} : -\frac{3}{128} : -\frac{3}{128}$ and $(s) \frac{3}{64} : \frac{3}{512} : -\frac{3}{512} : -\frac{3}{512}$. More examples will be given in [23]. We thus conjecture that the existence of these GR ratios of Eq.(2) in the RR persists to all orders in the Regge expansion of high energy string scattering amplitudes.

In conclusion, physically, the connection between Kummer function and high energy string scattering amplitudes derived in this letter may shed light on a deeper understanding of stringy symmetries. In contrast to an infinite number of ratios obtained in the GR previously, here one gets a nice Kummer function in the RR, which surely contains more analytic properties than just ratios for string scatterings. Mathematically, the proof of identity in Eq.(17) brings an interesting bridge between string theory and combinatoric number theory. Finally, in addition to the Kummer function in the leading order amplitudes discovered in this paper, the structure of the coefficient functions of the subleading order amplitudes in the RR maybe the most exciting mathematical problems to study.

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